REFLEXIVITY AND THE SUP OF LINEAR FUNCTIONALS

BY

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ABSTRACT

A relatively easy proof is given for the known theorem that a Banach space is reflexive if and only if each continuous linear functional attains its sup on the unit ball. This proof simplifies considerably for separable spaces and can be extended to give a proof that a bounded w-closed subset X of a complete locally convex linear topological space is w-compact if and only if each continuous linear functional attains its sup on X.

In 1950, it was proved [2] that a Banach space B is reflexive if B has a basis and each continuous linear functional attains its sup on the unit ball of each isomorph of B. The requirement that B have a basis was removed quickly by Klee [6]. Later it was shown that B is reflexive if each continuous linear functional attains its sup on the unit ball [3, p. 215, Theorem 5] and then [4, p. 139, Theorem 6] that a bounded w-closed subset X of a complete locally convex linear topological space is w-compact if each continuous linear functional attains its sup on X. Some streamlining and additional applications of the methods involved in the proof of this theorem have been given by Pryce [7] and Simons [10]. Some of the techniques that we use in the proofs of Theorems 1 and 2 were suggested by methods used in proving Lemma 2 of [9].

The statements and proofs of Theorems 1 and 2 are elementary. Equivalence of reflexivity and w-compactness of the unit ball is not used. The only concepts needed from the theory of Banach spaces are reflexivity, some elementary facts about continuous linear functionals, and Helly's condition [5, p. 151]: If $f_1, \cdots, f_n$ are linear functionals and $M, c_1, \cdots, c_n$ are scalars, then the following are equivalent:

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(a) $|\Sigma_1^n a_i c_i| \leqq M \| \Sigma_1^n a_i f_i \|$ for all $\{a_i\}$.

(b) For each $\varepsilon > 0$, there is an x such that $\| x \| < M + \varepsilon$ and

$$f_i(x) = c_i \quad \text{if } 1 \leqq i \leqq n.$$

Condition (ii) of Theorem 1 is known to be a sufficient condition for nonreflexivity [1, p. 58], but not a necessary condition [1, p. 109, (4)]. We let $V_n\{f_i\}$ denote the convex span of $\{f_i : i \geqq n\}$.

LEMMA 1. *Let B be a Banach space, θ a number in the interval $(0,1)$, and $\{f_n\}$ a sequence in the unit ball of B^* for which*

$$\|f\| \geqq \theta \ \text{ if } f \in \operatorname{conv}\{f_n\}.$$

If $\{\lambda_n\}$ is a sequence of positive numbers with $\Sigma_1^\infty \lambda_i = 1$, then there is a number α in the interval $[\theta, 1]$ and a sequence $\{g_n\}$ for which $g_n \in V_n\{f_i\}$ for each i,

$$\left\| \sum_1^\infty \lambda_i g_i \right\| = \alpha,$$

and, for each n,

$$\left\| \sum_1^n \lambda_i g_i \right\| < \alpha\left(1 - \theta \sum_{n+1}^\infty \lambda_i\right).$$

PROOF. Let $\{\varepsilon_n\}$ be a sequence of positive numbers for which

$$\sum_{k=1}^\infty \frac{\lambda_k \varepsilon_k}{\sum_{k+1}^\infty \lambda_i \sum_k^\infty \lambda_i} < 1 - \theta. \tag{1}$$

Choose $\{g_n\}$ inductively so that $g_n \in V_n\{f_i\}$ for each n and

$$\left\| \sum_1^{n-1} \lambda_i g_i + \left(\sum_n^\infty \lambda_i \right) g_n \right\| < \alpha_n(1 + \varepsilon_n), \tag{2}$$

where

$$\alpha_n = \inf\left\{ \left\| \sum_1^{n-1} \lambda_i g_i + \left(\sum_n^\infty \lambda_i \right) g \right\| : g \in V_n\{f_i\} \right\}.$$

Since g and each g_i are in the unit ball, $\theta \leqq \alpha_n \leqq 1$. Also,

$$\alpha_n \leqq \inf \left\| \left\{ \sum_1^{n-1} \lambda_i g_i + \lambda_n g_n + \left(\sum_{n+1}^\infty \lambda_i \right) g \right\| : g \in V_{n+1}\{f_i\} \right\} = \alpha_{n+1},$$

so that $\lim_{n\to\infty} \alpha_n = \alpha$ exists and $\theta \leqq \alpha = \| \Sigma_1^\infty \lambda_i g_i \| \leqq 1$. It follows from the triangle inequality and (2) that

$$\left\| \sum_1^n \lambda_i g_i \right\| \leqq \frac{\lambda_n}{\sum_n^\infty \lambda_i} \left\| \sum_1^{n-1} \lambda_i g_i + \left(\sum_n^\infty \lambda_i \right) g_n \right\| + \frac{\sum_{n+1}^\infty \lambda_i}{\sum_n^\infty \lambda_i} \left\| \sum_1^{n-1} \lambda_i g_i \right\|$$

$$< \left(\sum_{n+1}^\infty \lambda_i \right) \left[\frac{\lambda_n \alpha_n (1+\varepsilon_n)}{\sum_{n+1}^\infty \lambda_i \sum_n^\infty \lambda_i} + \frac{1}{\sum_n^\infty \lambda_i} \left\| \sum_1^{n-1} \lambda_i g_i \right\| \right] \tag{3}$$

Then successive applications of (3) for decreasing values of n give

$$\left\| \sum_1^n \lambda_i g_i \right\|$$

$$< \left(\sum_{n+1}^\infty \lambda_i \right) \left[\frac{\lambda_n \alpha_n (1+\varepsilon_n)}{\sum_{n+1}^\infty \lambda_i \sum_n^\infty \lambda_i} + \frac{\lambda_{n-1} \alpha_{n-1} (1+\varepsilon_{n-1})}{\sum_n^\infty \lambda_i \sum_{n-1}^\infty \lambda_i} + \frac{1}{\sum_{n-1}^\infty \lambda_i} \left\| \sum_1^{n-2} \lambda_i g_i \right\| \right]$$

$$\cdots\cdots\cdots$$

$$< \left(\sum_{n+1}^\infty \lambda_i \right) \left[\sum_{k=2}^n \frac{\lambda_k \alpha_k (1+\varepsilon_k)}{\sum_{k+1}^\infty \lambda_i \sum_k^\infty \lambda_i} + \frac{1}{\sum_2^\infty \lambda_i} \left\| \lambda_1 g_1 \right\| \right]$$

$$< \left(\sum_{n+1}^\infty \lambda_i \right) \left[\sum_{k=1}^n \frac{\lambda_k \alpha_k (1+\varepsilon_k)}{\sum_{k+1}^\infty \lambda_i \sum_k^\infty \lambda_i} \right].$$

Now we replace each α_k by α and use (1) to obtain

$$\left\| \sum_1^n \lambda_i g_i \right\| < \alpha \left(\sum_{n+1}^\infty \lambda_i \right) \left[\sum_{k=1}^n \left(\frac{1}{\sum_{k+1}^\infty \lambda_i} - \frac{1}{\sum_k^\infty \lambda_i} \right) + (1-\theta) \right]$$

$$= \alpha \left(1 - \theta \sum_{n+1}^\infty \lambda_i \right)$$

THEOREM 1. *If B is a* separable *Banach space, then the following are equivalent.*

(i) *B is not reflexive.*

(ii) *If $0 < \theta < 1$, then there is a sequence $\{f_n\}$ in the unit ball of B^* for which*

$$\| f \| \geqq \theta \text{ if } f \in \operatorname{conv}\{f_n\},$$

and $\lim_{n\to\infty} f_n(x) = 0$ *if* $x \in B$.

(iii) *If* $0 < \theta < 1$ *and* $\{\lambda_n\}$ *is a sequence of positive numbers with* $\sum_1^\infty \lambda_i = 1$, *then there is a number* α *in the interval* $[\theta, 1]$ *and a sequence* $\{g_n\}$ *for which* g_i *is in the unit ball of* B^*, $\lim_{n\to\infty} g_n(x) = 0$ *if* $x \in B$,

$$\left\| \sum_1^\infty \lambda_i g_i \right\| = \alpha,$$

and, for each n,

$$\left\| \sum_1^n \lambda_i g_i \right\| < \alpha\left(1 - \theta \sum_{n+1}^\infty \lambda_i\right) . \tag{4}$$

(iv) *There is a continuous linear functional that does not attain its* sup *on the unit ball of* B.

Proof. Suppose B is not reflexive and let F be a member of B^{**} for which $\|F\| < 1$ and

$$\operatorname{dist}(F, B^c) > \theta,$$

where B^c is the canonical image of B in B^{**}. Let $\{x_n\}$ be dense in B and for each n choose f_n such that

(a) $\|f_n\| < 1$,

(b) $F(f_n) = \theta$,

(c) $f_n(x_i) = 0$ if $i \leqq n$.

For Helly's condition to be applied to (b) and (c), with (c) written as $x_i^c(f_n) = 0$, we need

$$\theta \leqq M \left\| F + \sum_1^n a_i x_i^c \right\| \text{ for all } \{a_i\}.$$

Since this is satisfied if $M = \theta/\operatorname{dist}(F, B^c) < 1$, there exists f_n that satisfies (a) as well as (b) and (c). If $f \in \operatorname{conv}\{f_n\}$, then $F(f) = \theta$ and $\|f\| > \theta$. Clearly, $f_n(x) \to 0$ for each x in B.

To show that (ii) $\Rightarrow$ (iii), we need only use Lemma 1 and the observation that $\lim_{n\to\infty} g_n(x) = 0$ follows from $\lim_{n\to\infty} f_n(x) = 0$ and $g_n \in V_n\{f_i\}$ for each n.

To prove that (iii) $\Rightarrow$ (iv), we shall assume (iii) is satisfied and show that $\sum_1^\infty \lambda_i g_i$ does not attain its sup on the unit ball of B. Let x be an arbitrary member of the unit ball of B. Choose n so that $g_i(x) < \alpha\theta$ if $i > n$. Then

$$\sum_{1}^{\infty} \lambda_i g_i(x) < \sum_{1}^{n} \lambda_i g_i(x) + \alpha\theta \sum_{n+1}^{\infty} \lambda_i$$
$$< \left\| \sum_{1}^{n} \lambda_i g_i \right\| + \alpha\theta \sum_{n+1}^{\infty} \lambda_i,$$

and it follows from this and (4) that

$$\sum_{1}^{\infty} \lambda_i g_i(x) < \alpha\left(1 - \theta \sum_{n+1}^{\infty} \lambda_i\right) + \alpha\theta \sum_{n+1}^{\infty} \lambda_i = \alpha.$$

Thus the sup α of $\Sigma_1^\infty \lambda_i g_i$ on the unit ball is not attained at x.

To conclude the proof of Theorem 1, we need to show B is not reflexive if there is a continuous linear functional ϕ that does not attain its sup on the unit ball of B. By the Hahn-Banach theorem, there is a member F of B^{**} for which $\| F \| = 1$ and $F(\phi) = \| \phi \|$. If B is reflexive, there is an $x \in B$ such that $\| x \| = 1$ and $F(g) = g(x)$ for each $g \in B^*$. Then $F(\phi) = \phi(x) = \| \phi \|$, so ϕ attains its sup on the unit ball at x.

The next lemma and theorem are natural generalizations of Lemma 1 and Theorem 1 to general Banach spaces. For an arbitrary sequence of linear functionals $\{\phi_n\}$ on a linear space X, let $L\{\phi_n\}$ denote the set of all linear functionals ω which have the property that

$$\liminf_{n\to\infty} \phi_n(x) \leqq \omega(x) \leqq \limsup_{n\to\infty} \phi_n(x)$$

if $x \in X$. For example, ω satisfies this inequality if l is a Banach limit and $\omega(x) = l(\{\phi_n(x)\})$, or if l is any linear functional on l^∞ for which $\| l \| = 1$ and $l(x) = \lim x_n$ if $x = \{x_n\}$ and $\lim x_n$ exists.

LEMMA 2. *Let B be a Banach space, θ a number in the interval $(0, 1)$, and $\{f_n\}$ a sequence in the unit ball of B^* for which*

$$\|f - \omega\| \geqq \theta \text{ if } f \in \operatorname{conv}\{f_n\} \text{ and } \omega \in L\{f_i\}.$$

If $\{\lambda_i\}$ is a sequence of positive numbers with $\Sigma_1^\infty \lambda_i = 1$, then there is a number α in the interval $[0,2]$ and a sequence $\{g_i\}$ for which each g_i is in the unit ball of B^*, and, for each n and each linear functional ω in $L\{g_i\}$,

$$\left\| \sum_{1}^{\infty} \lambda_i(g_i - \omega) \right\| = \alpha \text{ and } \left\| \sum_{1}^{n} \lambda_i(g_i - \omega) \right\| < \alpha\left(1 - \theta \sum_{n+1}^{\infty} \lambda_i\right).$$

PROOF. Let $\{\varepsilon_n\}$ be a sequence of positive numbers for which

$$\sum_{k=1}^{\infty} \frac{\lambda_k \varepsilon_k}{\sum_{k+1}^{\infty} \lambda_i \sum_{k}^{\infty} \lambda_i} < 1 - \theta. \tag{5}$$

For each n and each sequence of linear functionals $\{\phi_i\}$, let $V_n\{\phi_i\}$ denote the convex span of $\{\phi_i : i \geqq n\}$ and let $V\{\phi_i\}$ denote the set of all sequences $\{\psi_i\}$ such that $\psi_n \in V_n\{\phi_i\}$ for each n. Note that if $\{\psi_i\} \in V\{\phi_i\}$, then $V\{\psi_i\} \subset V\{\phi_i\}$. For each n, let a number α_n, a linear functional g_n, and sequences $\{\phi_i^n\}$ and $\{\psi_i^n\}$ of continuous linear functionals be chosen inductively as follows. Suppose α_k, g_k, $\{\phi_i^k\}$ and $\{\psi_i^k\}$ have been chosen for $k < n$, where $n \geqq 1$. Let $\{\psi_i^0\} = \{f_i\}$ and

$$\alpha_n = \inf\left[\sup\left\{\left\|\sum_1^{n-1} \lambda_i g_i + \left(\sum_n^{\infty} \lambda_i\right) g - \omega\right\| : \omega \in L\{\phi_i\}\right\}\right], \tag{6}$$

where the inf is over all g and $\{\phi_i\}$ for which $g \in V_n\{\psi_i^{n-1}\}$ and $\{\phi_i\} \in V\{\psi_i^{n-1}\}$. Now choose $g_n \in V_n\{\psi_i^{n-1}\}$ and $\{\phi_i^n\} \in V\{\psi_i^{n-1}\}$ so that

$$\alpha_n \leqq \sup\left\{\left\|\sum_1^{n-1} \lambda_i g_i + \left(\sum_n^{\infty} \lambda_i\right) g_n - \omega\right\| : \omega \in L\{\phi_i^n\}\right\} < \alpha_n(1 + \varepsilon_n), \tag{7}$$

and then choose $\omega' \in L\{\phi_i^n\}$ so that

$$\alpha_n(1 - \varepsilon_n) < \left\|\sum_1^{n-1} \lambda_i g_i + \left(\sum_n^{\infty} \lambda_i\right) g_n - \omega'\right\| < \alpha_n(1 + \varepsilon_n).$$

Let $\bar{x}$ be a particular member of the unit ball of B such that

$$\alpha_n(1 - \varepsilon_n) < \sum_1^{n-1} \lambda_i g_i(\bar{x}) + \left(\sum_n^{\infty} \lambda_i\right) g_n(\bar{x}) - \omega'(\bar{x}). \tag{8}$$

Since $\liminf_{i \to \infty} \phi_i^n(\bar{x}) \leqq \omega'(\bar{x})$, there is a subsequence $\{\psi_i^n\}$ of $\{\phi_i^n\}$ such that, for each ω in $L\{\psi_i^n\}$,

$$\liminf_{i \to \infty} \phi_i^n(\bar{x}) = \lim_{i \to \infty} \psi_i^n(\bar{x}) = \omega(\bar{x}) \leqq \omega'(\bar{x}).$$

This implies (8) is satisfied if $\omega'(\bar{x})$ is replaced by $\omega(\bar{x})$. It follows from this, (7), and $L\{g_n\} \subset L\{\phi_i^n\}$, that

$$\alpha_n(1 - \varepsilon_n) < \left\|\sum_1^{n-1} \lambda_i g_i + \left(\sum_n^{\infty} \lambda_i\right) g_n - \omega\right\| < \alpha_n(1 + \varepsilon_n) \tag{9}$$

if $\omega \in L\{g_n\}$.

Since $\|g\| \leqq 1$ if $g \in \operatorname{conv}\{g_n\}$, and $\|\omega\| \leqq 1$ if $\omega \in L\{g_n\}$, we have $\alpha_n \leqq 2$ for each n. Observe that α_n in (6) does not decrease if the inf is taken over all g and $\{\phi_i\}$ for which $g \in V_n\{\psi_i^{n-1}\}$ and

$$g = \frac{\lambda_n}{\sum_n^\infty \lambda_i} g_n + \frac{\sum_{n+1}^\infty \lambda_i}{\sum_n^\infty \lambda_i} h \quad \text{with } h \in V_{n+1}\{\psi_i^n\},$$

and $\{\phi_i\} \in V\{\psi_i^n\}$. That is, $\alpha_n \leqq \alpha_{n+1}$, so that $\lim_{n\to\infty} \alpha_n = \alpha$ exists and $\theta \leqq \alpha = \| \sum_1^\infty \lambda_i g_i \| \leqq 2$.

It follows from the triangle inequality and (9) that, if $\omega \in L\{g_i\}$, then

$$\left\| \sum_1^n \lambda_i(g_i - \omega) \right\|$$

$$\leqq \frac{\lambda_n}{\sum_n^\infty \lambda_i} \left\| \sum_1^{n-1} \lambda_i(g_i - \omega) + \left(\sum_n^\infty \lambda_i \right)(g_n - \omega) \right\| + \frac{\sum_{n+1}^\infty \lambda_i}{\sum_n^\infty \lambda_i} \left\| \sum_1^{n-1} \lambda_i(g_i - \omega) \right\|$$

$$< \left(\sum_{n+1}^\infty \lambda_i \right) \left[\frac{\lambda_n \alpha_n (1 + \varepsilon_n)}{\sum_{n+1}^\infty \lambda_i \sum_n^\infty \lambda_i} + \frac{1}{\sum_n^\infty \lambda_i} \left\| \sum_1^{n-1} \lambda_i(g_i - \omega) \right\| \right]. \tag{10}$$

Then successive applications of (10) for decreasing values of n give

$$\left\| \sum_1^n \lambda_i(g_i - \omega) \right\|$$

$$< \left(\sum_{n+1}^\infty \lambda_i \right) \left[\frac{\lambda_n \alpha_n (1 + \varepsilon_n)}{\sum_{n+1}^\infty \lambda_i \sum_n^\infty \lambda_i} + \frac{\lambda_{n-1} \alpha_{n-1} (1 + \varepsilon_{n-1})}{\sum_n^\infty \lambda_i \sum_{n-1}^\infty \lambda_i} + \frac{1}{\sum_{n-1}^\infty \lambda_i} \left\| \sum_1^{n-2} \lambda_i(g_i - \omega) \right\| \right]$$

$$\cdots\cdots\cdots\cdots$$

$$< \left(\sum_{n+1}^\infty \lambda_i \right) \left[\sum_{k=2}^n \frac{\lambda_k \alpha_k (1 + \varepsilon_k)}{\sum_{k+1}^\infty \lambda_i \sum_k^\infty \lambda_i} + \frac{1}{\sum_2^\infty \lambda_i} \| \lambda_1 (g_1 - \omega) \| \right]$$

$$< \left(\sum_{n+1}^\infty \lambda_i \right) \left[\sum_{k=1}^n \frac{\lambda_k \alpha_k (1 + \varepsilon_k)}{\sum_{k+1}^\infty \lambda_i \sum_k^\infty \lambda_i} \right].$$

Now we replace each α_k by α and use (5) to obtain

$$\left\| \sum_{1}^{n} \lambda_i(g_i-\omega)\right\| < \alpha \left(\sum_{n+1}^{\infty} \lambda_i\right) \left[\sum_{k=1}^{n} \left(\frac{1}{\sum_{k+1}^{\infty} \lambda_i} - \frac{1}{\sum_{k}^{\infty} \lambda_i}\right) + (1-\theta)\right]$$

$$= \alpha\left(1 - \theta \sum_{n+1}^{\infty} \lambda_i\right).$$

THEOREM 2. *If B is a Banach space, then the following are equivalent.*

(i) *B is not reflexive.*

(ii) *If $0 < \theta < 1$, then there is a subspace X of B and a sequence $\{f_n\}$ in the unit ball of B^* for which*

$$\|f-\omega\| \geqq \theta \quad \textit{if } f \in \operatorname{conv}\{f_n\} \textit{ and } \omega \in X^{\perp},$$

and $\lim_{n\to\infty} f_n(x) = 0$ *if* $x \in X$.

(iii) *If $0 < \theta < 1$ and $\{\lambda_i\}$ is a sequence of positive numbers with $\Sigma_1^\infty \lambda_i = 1$, then there is a number α in the interval $[\theta, 2]$ and a sequence $\{g_i\}$ for which each g_i is in the unit ball of B^*, and, for each n and each linear functional ω in $L\{g_i\}$,*

$$\left\| \sum_{1}^{\infty} \lambda_i(g_i - \omega)\right\| = \alpha \textit{ and } \left\| \sum_{1}^{n} \lambda_i(g_i - \omega)\right\| < \alpha\left(1 - \theta \sum_{n+1}^{\infty} \lambda_i\right). \tag{11}$$

(iv) *There is a continuous linear functional that does not attain its* sup *on the unit ball of B.*

PROOF. Let X be a nonreflexive separable subspace of B. Then it follows from Theorem 1 that there is a sequence $\{f_n\}$ with each f_n in the unit ball of B^*, $\lim_{n\to\infty} f_n(x) = 0$ if $x \in X$, and $\|f\|_X \geqq \theta$ if $f \in \operatorname{conv}\{f_n\}$, where $\|f\|_X$ is the norm of f as a linear functional on X. Then

$$\|f-\omega\| \geqq \|f-\omega\|_X = \|f\|_X \geqq \theta \quad \text{if } f \in \operatorname{conv}\{f_n\} \text{ and } \omega \in X^{\perp}.$$

To show that (ii) $\Rightarrow$ (iii), we need only use Lemma 2 and the observation that if $\{f_n\}$, θ and X are as described in (ii), then $\omega \in X^{\perp}$ if $\omega \in L\{f_n\}$.

To prove that (iii) $\Rightarrow$ (iv), we shall assume that (iii) is satisfied and choose the numbers in (iii) so that there is a number Δ such that $0 < \Delta < \frac{1}{2}\theta^2$ and, for each n,

$$\lambda_{n+1} < \Delta\lambda_n. \tag{12}$$

We shall then show that $\Sigma_1^\infty \lambda_i(g_i - \omega)$ does not attain its sup on the unit ball of B if $\omega \in L\{g_i\}$. Let x be an arbitrary member of the unit ball of B. Since $\liminf_{i\to\infty} g_i(x) \leqq \omega(x)$ and $\theta \leqq \alpha$, there is an n such that

$$(g_{n+1}-\omega)(x) < \theta^2 - 2\Delta \leqq \alpha\theta - 2\Delta.$$

Then

$$\begin{aligned}\sum_1^\infty \lambda_i(g_i-\omega)(x) &< \sum_1^n \lambda_i(g_i-\omega)(x) + (\alpha\theta-2\Delta)\lambda_{n+1} + \sum_{n+2}^\infty \lambda_i(g_i-\omega)(x)\\ &\leqq \left\| \sum_1^n \lambda_i(g_i-\omega)\right\| + (\alpha\theta-2\Delta)\lambda_{n+1} + 2\sum_{n+2}^\infty \lambda_i,\end{aligned}$$

and it follows from this and (11) that

$$\sum_1^\infty \lambda_i(g_i-\omega)(x) < \alpha\left(1-\theta\sum_{n+1}^\infty \lambda_i\right) + (\alpha\theta-2\Delta)\lambda_{n+1} + 2\sum_{n+2}^\infty \lambda_i.$$

From (12), we have $\sum_{n+2}^\infty \lambda_i < \Delta\sum_{n+1}^\infty \lambda_i$, so that

$$\begin{aligned}\sum_1^\infty \lambda_i(g_i-\omega)(x) &< \alpha - (\alpha\theta-2\Delta)\sum_{n+1}^\infty \lambda_i + (\alpha\theta-2\Delta)\lambda_{n+1}\\ &= \alpha - (\alpha\theta-2\Delta)\sum_{n+2}^\infty \lambda_i\\ &< \alpha.\end{aligned}$$

Thus the sup α of $\sum_1^\infty \lambda_i(g_i-\omega)(x)$ on the unit ball is not attained at x.

Exactly as in the proof of Theorem 1, (iv) implies nonreflexivity.

The next two theorems are proved by reducing them to the Banach space case and then using methods similar to those used in Theorems 1 and 2. We let $S_X(f)$ denote sup $\{f(x): x \in X\}$.

THEOREM 3. *If X is a* separable *bounded w-closed subset of a complete locally convex linear topological space, then the following are equivalent.*

(i) *X is not w-compact.*

(ii) *There is a positive number θ and a sequence of equicontinuous linear functionals $\{f_n\}$ for which*

$$S_X(|f|) \geqq \theta \quad \text{if } f \in \operatorname{conv}\{f_n\}, \tag{13}$$

and $\lim_{n\to\infty} f_n(x) = 0$ *if* $x \in X$.

(iii) *There is a positive number θ such that, if $\{\lambda_i\}$ is a sequence of positive numbers with $\sum_1^\infty \lambda_i = 1$, then there is a number $\alpha \geqq \theta$ and a sequence $\{g_i\}$ of equicontinuous linear functionals for which* $\lim_{n\to\infty} g_n(x) = 0$ *if* $x \in X$,

$$S_X\left(\left|\sum_1^\infty \lambda_i g_i\right|\right) = \alpha$$

and, for each n,

$$S_X\left(\left|\sum_1^n \lambda_i g_i\right|\right) < \alpha\left(1 - \theta \sum_{n+1}^{\infty} \lambda_i\right).$$

(iv) *There is a continuous linear functional that does not attain its* sup *on X.*

PROOF. If T is a locally convex topological space, then T is isomorphic with a subspace of a product $\prod B_\alpha$ of Banach spaces [8, p. 54, Corollary 2]. In fact, if $\{p_\alpha\}$ is a generating family of semi-norms for T, then we can let $x \leftrightarrow \{x_\alpha\}$, where x_α is the member of $T/p_\alpha^{-1}(0)$ that contains x. If T is complete, then T is closed in $\prod B_\alpha$. Since T also is convex, T then is w-closed in $\prod B_\alpha$. The canonical projection X_α of X into B_α is bounded in B_α. If the weak closure of X_α as a subset of B_α were w-compact for each α, it would follow from the Tychonoff theorem that $\prod[\mathrm{wcl}(X_\alpha)]$ is compact in the product topology when each B_α is given its weak topology. This would imply X is w-compact, since X is w-closed in T and T is w-closed in $\prod B_\alpha$. Now choose α so that $\mathrm{wcl}(X_\alpha)$ is not w-compact in B_α. Since $\mathrm{wcl}(X_\alpha) \subset \mathrm{cl}[\mathrm{conv}(X_\alpha)]$, $\mathrm{wcl}(X_\alpha)$ is separable.

To complete the proof that (i) $\Rightarrow$ (ii), it now is sufficient to give a proof for the case where X is a separable bounded w-closed subset of a Banach space B. Suppose X is not w-compact. Let $Y = \mathrm{cl}[\mathrm{lin}(X)]$. Let C be the normed linear space of continuous linear functionals f on Y, with $S_X(|f|)$ as the norm of f. Then X, with the weak topology $w(X, Y^*)$, is homeomorphic to the natural image X^c of X in C^*. with the topology $w(C^*, C)$. Also, for x^c as a member of C^*,

$$|x^c(f)| = |f(x)| \leqq S_X(|f|),$$

so $\|x^c\| \leqq 1$ and X^c is contained in the unit ball of C^*. It follows from the Tychonoff theorem that the unit ball of C^* is compact when given the topology $w(C^*, C)$. If $Y^c \cap C^* = C^*$, then the inverse canonical image of the unit ball of C^* is w-compact and X is w-compact. Therefore there is an F in C^* that does not belong to Y^c. If $\sup\{\|x\| : x \in X\} = M$ and $\|F\|$ and $\|F\|_{C^*}$ are the norms of F as a member of Y^{**} and as a member of C^*, respectively, then

$$\|F\| \leqq M\|F\|_{C^*},$$

so $F \in Y^{**}$. Since Y is complete, Y^c is closed in Y^{**}. Let Δ be a positive number for which

$$\mathrm{dist}(F, Y^c) > \Delta,$$

where "dist" is relative to the norm of Y^{**}. Let $\{x_n\}$ be dense in X and for each n choose f_n such that

(a) $\|f_n\| < 1$,

(b) $F(f_n) = \Delta$,

(c) $f_n(x_i) = 0$ if $i \leqq n$.

For Helly's condition to be applied to (b) and (c), with (c) written as $x_i^c(f_n) = 0$ we need

$$\Delta \leqq M \left\| F + \sum_1^n a_i x_i^c \right\| \text{ for all } \{a_i\}.$$

Since this is satisfied if $M = \Delta/\mathrm{dist}(F, Y^c) < 1$, there is an f_n that satisfies (a) as well as (b) and (c). If $f \in \mathrm{conv}\{f_n\}$, then $F(f) = \Delta$ and

$$\Delta \leqq \| F \|_{C^*} S_X(|f|).$$

Thus if $\theta = \Delta / \| F \|_{C^*}$, then (13) is satisfied.

The proof that (ii) $\Rightarrow$ (iii) is obtained from the corresponding part of the proof of Theorem 1 (and Lemma 1) by replacing $\| \cdot \|$ by $S_X(| \cdot |)$ whenever it occurs and using equicontinuity of $\{f_n\}$ and boundedness of X to get a number β such that $\theta \leqq \alpha_n \leqq \beta$ (instead of $\theta \leqq \alpha_n \leqq 1$) and $\theta \leqq \alpha \leqq \beta$.

The proof that (iii) $\Rightarrow$ (iv) is obtained similarly, with the sign of each λ_i changed if necessary so that

$$S_X\left(\sum_1^\infty \lambda_i g_i \right) = \alpha$$

and, for each n,

$$S_X\left(\sum_1^n \lambda_i g_i \right) < \alpha\left(1 - \theta \sum_{n+1}^\infty |\lambda_i| \right).$$

To conclude the proof of Theorem 3, we need only observe that X is not w-compact if there is a continuous linear functional that does not attain its sup on X.

THEOREM 4. *If X is a bounded w-closed subset of a complete locally convex linear topological space, then the following are equivalent.*

(i) *X is not w-compact.*

(ii) *There is a positive number θ, a subset X_0 of X, and a sequence of equicontinuous linear functionals $\{f_n\}$ for which*

$$S_X(|f - \omega|) \geqq \theta \quad \textit{if } f \in \mathrm{conv}\{f_n\} \textit{ and } \omega \in X_0^\perp,$$

and $\lim_{n\to\infty} f_n(x) = 0$ *if* $x \in X_0$.

(iii) *There is a positive number* θ *such that, if* $\{\lambda_i\}$ *is a sequence of positive numbers with* $\sum_1^\infty \lambda_i = 1$, *then there is a positive number* α *and a sequence* $\{g_i\}$ *of equicontinuous linear functionals such that, for each n and each linear functional* ω *in* $L\{g_i\}$,

$$S_X\left(\left|\sum_1^\infty \lambda_i(g_i-\omega)\right|\right)=\alpha \text{ and } S_X\left(\left|\sum_1^n \lambda_i(g_i-\omega)\right|\right)<\alpha\left(1-\theta\sum_{n+1}^\infty \lambda_i\right).$$

(iv) *There is a continuous linear functional that does not attain its* sup *on* X.

Proof. Since a w-closed bounded subset X of a complete locally convex linear topological space is w-compact if and only if it is weakly countably compact [1, p. 51, Corollary 2], X is w-compact if and only if each w-closed separable subspace of X is w-compact. Thus the implication (i) $\Rightarrow$ (ii) follows from Theorem 3.

As for Theorem 3, the proof that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) need be done only for Banach spaces and can be obtained from the corresponding parts of the proofs of Lemma 2 and Theorem 2 by replacing $\|\cdot\|$ by $S_X(|\cdot|)$ and "the unit ball of B" by X, whenever they occur, using equicontinuity of $\{f_n\}$ and boundedness of X to get a number β such that $\theta \leqq \alpha_n \leqq \beta$, and observing that if θ, X_0, X, and $\{f_n\}$ are as described in (ii), then $\omega \in X_0^\perp$ if $\omega \in L\{f_n\}$.

To conclude the proof of Theorem 4, we observe again that X is not w-compact if there is a continuous linear functional that does not attain its sup on X.

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